Overall properties of planar quasisymmetric randomly inhomogeneous media: Estimates and cell models

Pham Duc Chinh
National Center for Natural Science and Technology, Vien Co hoc, 224 Doi Can, Hanoi, Vietnam
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We study the class of planar isotropic randomly inhomogeneous media with certain statistical symmetry among the component geometries. Exact upper and lower estimates of the conductivity and elastic properties for the whole class of multicomponent media are given explicitly in the properties and volume fractions of the constituents and are compared with some idealistic but exact cell models. The comparisons reveal that the estimates are attained, or nearly reached, by envelopes of exact properties of the models, hence the estimates should give reasonable approximations for the overall properties of the quasisymmetric mixtures, as well as the expected scatter intervals associated with the uncertainty in the geometry of the media. Special attention is given to very simple estimates for the properties of multicomponent circular cell media, which are expected to represent practical particulate composites. [S1063-651X(97)00807-6]

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I. INTRODUCTION

Many heterogeneous materials appear statistically homogeneous and isotropic and have overall (effective) properties depending upon the properties and volume fractions of the constituents and their geometry. The microscopic sizes of the constituents are often large in comparison with the atom dimensions and their behavior can be described by linear relationships from continuum physics, such as those between the fluxes and driving forces and strains and stresses. The contacts between the components in a heterogeneous medium are assumed to be ideal. Overall properties of such composite systems have been of interest since the time of Maxwell [1], Rayleigh [2], and Einstein [3]. Those authors obtained the asymptotic expressions of the effective dielectric (and resistivity) coefficient, and the viscosity for a dilute suspension of spherical particles in a continuous matrix. The direct approach to the problem is to define a particular microgeometry and then proceed to solve the field equations in this geometry. However, such exact solutions are rarely available because the microgeometry of a composite is often of random nature and the inhomogeneities are not of idealistic spherical form or presented in a dilute fraction. Hence, various effective medium approximate schemes have been constructed to deal with the problem (see Refs. [4–17] and references therein). The advantage of these schemes is that they give relatively simple evaluations of the effective properties expressed in the properties and volume fractions of the constituents and they might explain the main effects of interactions between the constituents in appropriate situations. However, the schemes cannot tell the degree of accuracy of the obtained results and they sometimes violate the exact relations imposed upon them, e.g., the expressions of the effective properties should be attained by definite specific models. Perhaps, a more complete approach begun in the pioneer works [5,18], and followed in Refs. [19–36], is to derive upper and lower bounds on the effective properties from dual variational principles of the respective problems. The estimates should yield not only the approximate effective properties of a composite, but also the possible scatter intervals associated with the uncertainty in the composite geometry. The difficult point, however, is to extract usable information about restrictions on the geometry of a given composite and then derive the best possible estimates of the effective properties—the ones that can be attained by specific geometries permitted by the given uncertainty in the composite geometry. In recent papers (Refs. [29–31, 33–35]) we developed a variational approach to the problem. The approach constructs trial polarization fields similar to those of Hashin and Shtrikman [22], however our inequalities keep additional fluctuation terms, which help to tighten the bounds considerably.

Returning to the effective medium approximation methods, we are especially interested in the symmetric self-consistent scheme of Bruggeman [7], Landauer [8,14], Budiansky [10], and Hill [11]. Those authors calculated the fields in the heterogeneities by considering them separately and equally as spherical (or more generally ellipsoidal) particles imbeded in a homogeneous medium with an unknown effective property and then proceed to construct and solve the self-consistent equation determining the effective property. Landauer [8] and Budiansky [10] argue that the method is reasonable for those mixtures made of the constituents combined in a certain symmetrical fashion, while the method predicts inaccurate results for certain asymmetric matrix composites. It should be noted that a self-consistent effective property is a reliazable one and corresponds to a definite differential geometric cell model [7,37,38] constructed as follows: starting with some base matrix, at each step of the procedure, we add into it infinitesimal amounts of well separated new spherical particles (or randomly oriented ellipsoidal ones with the same aspect ratio) of inclusion phases, with relative proportions corresponding to their volume fractions in the final composition. The inclusions added at each step must be considerably greater in sizes than those that have been added previously, and they will see an effective continuum, owing to their relative sizes. The procedure is continued until the volume fraction of the original base matrix becomes infinitely small and eliminated. The polydispersed

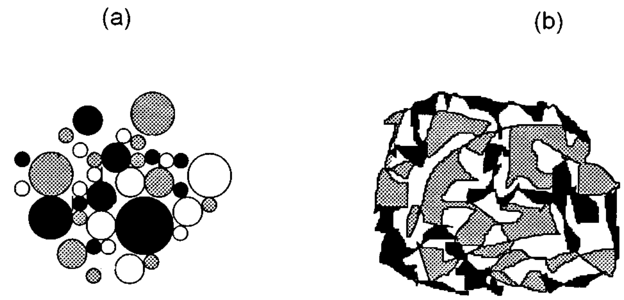

FIG. 1. (a) Circular cell material; (b) quasisymmetric material.

differential model constructed possesses an effective property exactly equal to that determined by the symmetric self-consistent scheme. Clearly the model is only a special member of the class of quasisymmetric cell composites [21,33,36] formed from disordered spherical cells (or randomly oriented ellipsoidal ones with the same aspect ratio) having sizes ranging to infinitely small, to fill all the material space and the properties assigned randomly with the proportions corresponding to the volume fractions of the phases in the composition [consult Fig. 1(a)]. These idealistic mixtures, in their turn, belong to a larger but realistic class of quasisymmetric composites [21,30,34–36] [referred to in those works as symmetric cell, fully disordered or perfectly random materials—see Fig. 1(b)]. Such composites generally do not have distinct inclusion and matrix phases as well as distinct forms or sizes of the heterogeneities, the microgeometries of their components are statistically indistinguishable, except possible differences in the volume fractions. In the special case of equal volume fractions of the phases, we have a truly symmetric composite: an interchange of the places between any two phases should not affect the overall properties of the composite. Upper and lower estimates for the effective properties of those quasisymmetric composites will be the subject of our study. We also restrict our attention in this paper to those that are globally and locally isotropic in two dimensions, which correspond to the behavior of certain unidirectional composites in the transverse plane and of the thin films. Among the quasisymmetric multicomponent media, the circular cell composite, which is the simplest model representing practical equiaxial particulate mixtures, will be of our primary interest.

II. BOUNDS ON THE CONDUCTIVITY

Consider a representative element V of a multicomponent medium, which consists of n isotropic components occupying regions $V_\alpha \subset V$ of volumes v_α (the volume of V is assumed to be the unity) and having the conductivities λ_α, elastic moduli k_α, μ_α $(\alpha = 1, \ldots, n)$. As the composite is statistically isotropic, its effective conductivity λ_c can be defined as follows:

$$\lambda_c \mathbf{e}^0 \cdot \mathbf{e}^0 = \inf \int_V \lambda \mathbf{e} \cdot \mathbf{e} \, d\mathbf{x}, \tag{1}$$

where

$$\lambda(\mathbf{x}) = \sum_\alpha \lambda_\alpha \kappa_\alpha(\mathbf{x}), \quad \kappa_\alpha(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in V_\alpha \\ 0, & \mathbf{x} \notin V_\alpha . \end{cases} \tag{2}$$

The infimum (1) is seen over the vector field $e(x)$ (electrical, magnetic, thermal, . . .) satisfying restrictions

$$\text{rot } \mathbf{e} = \mathbf{0} \text{ (i.e., } \mathbf{e} \text{ is a gradient field)},$$

$$\int_V \mathbf{e} \, d\mathbf{x} = \mathbf{e}^0 = \text{const}. \tag{3}$$

The exact infimum point $e(x)$ of problems (1), (3) should yield an exact solution for λ_c. However, such an exact solution generally is not available because of the statistical nature of composite geometry. So our objective is to find the best possible trial field $e(x)$ to deduce the best possible upper bound on λ_c from Eq. (1), given the degree of uncertainty in the microgeometry of V.

Following the established procedure of Refs. [29, 34], we substitute a trial field $e(x)$ with the components

$$e_i = e_i^0 - 2 \sum_{\alpha=1}^{n} \left[1 - \left(\sum_{\beta=1}^{n} v_\beta \frac{\lambda_\alpha + \lambda_0}{\lambda_\beta + \lambda_0} \right)^{-1} \right] e_j^0 \varphi^\alpha_{,ij}, \tag{4}$$

into Eq. (1) to deduce the upper bound

$$\lambda_c \leqslant P_\lambda(\lambda_0) + \lambda_{**}, \tag{5}$$

where

$$P_\lambda(\lambda_0) = \left(\sum_{\alpha=1}^{n} \frac{v_\alpha}{\lambda_\alpha + \lambda_0} \right)^{-1} - \lambda_0, \tag{6}$$

$$\lambda_{**} = 2 \left(\sum_{\alpha=1}^{n} \frac{v_\alpha}{\lambda_\alpha + \lambda_0} \right)^{-2} \times \sum_{\alpha,\beta,\gamma=1}^{n} [(\lambda_\alpha - \lambda_0) A_\alpha^{\beta\gamma} X_\beta(\lambda_0) X_\gamma(\lambda_0)], \tag{7}$$

$$X_\beta(\lambda_0) = \sum_{\alpha=1}^{n} \frac{v_\alpha}{\lambda_\alpha + \lambda_0} - \frac{1}{\lambda_\beta + \lambda_0}, \tag{8}$$

$$A_\alpha^{\beta\gamma} = \int_{V_\alpha} \varphi^\beta_{ij} \varphi^\gamma_{ij} d\mathbf{x}, \quad \nabla^2 \varphi^\alpha(\mathbf{x}) = \kappa_\alpha(\mathbf{x}), \tag{9}$$

$$\varphi^\beta_{ij}(\mathbf{x}) = \varphi^\beta_{,ij} - \frac{1}{v_\alpha} \int_{V_\alpha} \varphi^\beta_{,ij} d\mathbf{x}, \quad \mathbf{x} \in V_\alpha ; \tag{10}$$

the latin indices after a comma denote differentiation with respect to the corresponding Cartesian coordinates; conventional summation on repeated latin subscripts is assumed.

The bounds (5)–(10) contain geometric parameters $A_\alpha^{\beta\gamma}$, which depend on the harmonic potentials φ^α defined on the regions V_α. If these geometric parameters can be evaluated for specific composites then the parameter λ_0, which is arbitrarily positive, should be chosen as small as possible to make $\lambda_{**} \leqslant 0$, so that the term λ_{**} in Eq. (5) can be eliminated to yield us the best possible upper bound on λ_c for the composites considered. For example, taking $\lambda_0 = \lambda_{\max}$

$=\max\{\lambda_1,...,\lambda_n\}$, one can verify that $\lambda_{**}\leqslant 0$ independent of particular values of $A_\alpha^{\beta\gamma}$, and from Eq. (5) one rederives the Hashin-Shtrikman upper bound [22] valid for all isotropic composites

$$\lambda_c \leqslant P_\lambda(\lambda_{\max}). \tag{11}$$

For the class of quasisymmetric composites [31], the parameters $A_\alpha^{\beta\gamma}$ can be shown to depend only on two positive coefficients $f_1, f_2 (\alpha\neq\beta\neq\gamma\neq\alpha)$

$$\begin{aligned} A_\alpha^{\beta\gamma} &= v_\alpha v_\beta v_\gamma (f_1 - f_2), \\ A_\alpha^{\alpha\alpha} &= v_\alpha(1-v_\alpha)[(1-v_\alpha)f_1 + v_\alpha f_2], \\ A_\alpha^{\alpha\beta} &= v_\alpha v_\beta[(v_\alpha - 1)f_1 - v_\alpha f_2], \\ A_\alpha^{\beta\beta} &= v_\alpha v_\beta[(1-v_\beta)f_2 + v_\beta f_1]. \end{aligned} \tag{12}$$

From Eqs. (5)–(7) and (12) one can deduce an upper bound for quasisymmetric composites, which is valid for all possible positive coefficients f_1, f_2

$$\lambda_c \leqslant P_\lambda(\lambda^u),$$

$$\lambda^u = \min\left\{ \lambda_0 \,\middle|\, \lambda_0 \geqslant \sum_{\alpha=1}^n v_\alpha \lambda_\alpha, Q_\lambda^u(\lambda_0) \leqslant 0 \right\},$$

$$Q_\lambda^u(\lambda_0) = \sum_{\alpha=1}^n v_\alpha(\lambda_\alpha - \lambda_0)\left(\sum_{\beta=1}^n \frac{v_\beta}{\lambda_\beta+\lambda_0} - \frac{1}{\lambda_\alpha+\lambda_0}\right)^2. \tag{13}$$

In the case of two-component materials, the bound (13) reduces further to

$$\lambda_c \leqslant P_\lambda(\lambda^u), \quad \lambda^u = \max\{v_1\lambda_1 + v_2\lambda_2, v_1\lambda_2 + v_2\lambda_1\}. \tag{14}$$

If the geometric coefficients f_1, f_2 can be evaluated for certain cell composites [21,33] then an even tighter bound can be constructed

$$\lambda_c \leqslant P_\lambda(\lambda^{uf}), \tag{15}$$

where u^{uf} is the solution of the equation (to make $\lambda_{**}=0$)

$$\begin{aligned} & f_1 \sum_\alpha v_\alpha(\sigma_\alpha - \sigma^{uf})\left(\sum_\beta \frac{v_\beta}{\sigma_\beta + \sigma^{uf}} - \frac{1}{\sigma_\alpha+\sigma^{uf}}\right)^2 \\ & + f_2 \sum_\alpha v_\alpha(\sigma_\alpha - \sigma^{uf}) \\ & \times \sum_\beta v_\beta \left(\sum_\gamma \frac{v_\gamma}{\sigma_\gamma+\sigma^{uf}} - \frac{1}{\sigma_\beta + \sigma^{uf}}\right)^2 = 0. \end{aligned} \tag{16}$$

Of particular interest are circular cell materials, for them, as in the respective case of three-dimensional spherical cell materials [33], we have $f_1=0$, so from Eqs. (15) and (16) we get a very simple estimate

$$\lambda_c \leqslant P_\lambda(\lambda^{uc}), \quad \lambda^{uc} = \sum_{\alpha=1}^n v_\alpha \lambda_\alpha. \tag{17}$$

Similarly one can construct the lower bound from the problem dual to that of Eqs. (1) and (3)

$$\lambda_c^{-1}\mathbf{j}^0\cdot\mathbf{j}^0 = \inf \int_V \lambda^{-1}\mathbf{j}\cdot\mathbf{j}\, d\mathbf{x}, \tag{18}$$

where the flux vector field $\mathbf{j(x)}$ is subjected to the restrictions

$$\nabla\cdot\mathbf{j} = 0 \quad (\text{equilibrium equation}),$$

$$\langle \mathbf{j}\rangle = \mathbf{j}^0 = \text{const.} \tag{19}$$

To construct a lower bound on λ_c (i.e., an upper bound on λ_c^{-1}) from Eqs. (18) and (19), we take a trial field $\mathbf{j(x)}$ with the components

$$j_i = j_i^0 - 2\sum_{\alpha=1}^n \left[1 - \left(\sum_{\beta=1}^n v_\beta \frac{\lambda_\alpha^{-1} + \hat\lambda_0^{-1}}{\lambda_\beta^{-1}+\hat\lambda_0^{-1}} \right)^{-1} \right] (j_j^0 \varphi_{,ij}^\alpha - j_i^0 \kappa_\alpha) \tag{20}$$

and then deduce from Eq. (18) the lower bound for λ_c

$$\lambda_c^{-1} \leqslant P_\lambda^{-1}(\hat\lambda_0) + \hat\lambda_{**}, \tag{21}$$

$$\begin{aligned} \hat\lambda_{**} &= 2\hat\lambda_0^2 \left(1 - \hat\lambda_0 \sum_{\alpha=1}^n \frac{v_\alpha}{\lambda_\alpha + \hat\lambda_0} \right)^{-2} \\ & \times \sum_{\alpha,\beta,\gamma=1}^n [(\lambda_\alpha^{-1} - \hat\lambda_0^{-1}) A_\alpha^{\beta\gamma} X_\beta(\hat\lambda_0) X_\gamma(\hat\lambda_0)]. \end{aligned} \tag{22}$$

Let $\hat\lambda_0$, which is arbitrarily positive, to take the value $\hat\lambda_0 = \lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_n\}$, one can verify that $\hat\lambda_{**} \leqslant 0$ independent of particular values of $A_\alpha^{\beta\gamma}$, then $\hat\lambda_{**}$ in Eq. (21) can be eliminated to strengthen the inequality leading to the Hashin-Shtrikman lower bound valid for all isotropic composites

$$\lambda_c \geqslant P_\lambda(\lambda_{\min}). \tag{23}$$

For our smaller class of quasisymmetric composites we have, respectively, the tighter bound

$$\lambda_c \geqslant P_\lambda(\lambda^l),$$

$$\lambda^l = \max\left\{ \hat\lambda_0 \,\middle|\, \hat\lambda_0 \leqslant \left(\sum_{\alpha=1}^n v_\alpha/\lambda_a \right)^{-1}, Q_\lambda^l(\hat\lambda_0) \leqslant 0 \right\},$$

$$Q_\lambda^l(\hat\lambda_0) = \sum_{\alpha=1}^n v_\alpha(1/\lambda_\alpha - 1/\hat\lambda_0)\left(\sum_{\beta=1}^n \frac{v_\beta}{\lambda_\beta + \hat\lambda_0} - \frac{1}{\lambda_\alpha + \hat\lambda_0}\right)^2. \tag{24}$$

For two-component materials Eq. (24) reduces to

$$\lambda_c \geqslant P_\lambda(\lambda^l),$$
$$\lambda^l = \min\{(v_1/\lambda_1 + v_2/\lambda_2)^{-1}, (v_1/\lambda_2 + v_2/\lambda_1)^{-1}\}. \tag{25}$$

For those quasisymmetric composites with given parameters f_1, f_2 a tighter bound can be derived

$$\lambda_c \geqslant P_\lambda(\lambda^{lf}), \tag{26}$$

where λ^{lf} is the solution of the equation (to make $\hat{\lambda}_{**}=0$)

$$\begin{aligned} &f_1 \sum_{\alpha=1}^{n} v_\alpha \left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_0^{lf}} \right) \left(\sum_{\beta=1}^{n} \frac{v_\beta}{\sigma_\beta + \sigma_0^{lf}} - \frac{1}{\sigma_\alpha + \sigma_0^{lf}} \right)^2 \\ &+ f_2 \sum_{\alpha=1}^{n} v_\alpha \left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_0^{lf}} \right) \\ &\times \sum_{\beta=1}^{n} v_\beta \left(\sum_{\gamma=1}^{n} \frac{v_\gamma}{\sigma_\gamma + \sigma_0^{lf}} - \frac{1}{\sigma_\beta + \sigma_0^{lf}} \right)^2 = 0. \end{aligned} \tag{27}$$

For circular cell materials the bound reduces to

$$\lambda_c \geqslant P_\lambda(\lambda^{lc}), \quad \lambda^{lc} = \left(\sum_{\alpha=1}^{n} v_\alpha \lambda_\alpha^{-1} \right)^{-1}. \tag{28}$$

Beran and Silnutzer [23] obtained the following bounds for the quasisymmetric two-component cell materials:

$$B_\lambda^u \geqslant \lambda_c \geqslant B_\lambda^l, \tag{29}$$

$$B_\lambda^u = v_1\lambda_1 + v_2\lambda_2 - \frac{v_1 v_2}{2} \frac{(\lambda_1-\lambda_2)^2}{v_1\lambda_1 + v_2\lambda_2 + 2G(v_2^2 - v_1^2)(\lambda_1-\lambda_2)},$$

$$B_\lambda^l = \left[\frac{v_1}{\lambda_1} + \frac{v_2}{\lambda_2} - \frac{v_1 v_2}{2\lambda_1\lambda_2} \frac{(\lambda_1-\lambda_2)^2}{v_1\lambda_2 + v_2\lambda_1 + 2G(v_1^2 - v_2^2)(\lambda_1-\lambda_2)} \right]^{-1}. \tag{30}$$

The geometric parameter G in Eq. (30) is restricted by the inequality $1/4 \leqslant G \leqslant 1/2$, so for the whole class of quasisymmetric materials we have

$$\max_{1/4\leqslant G\leqslant 1/2} B_\lambda^u \geqslant \lambda_c \geqslant \min_{1/4\leqslant G\leqslant 1/2} B_\lambda^l. \tag{31}$$

The bounds (31) are identical to those of Eqs. (14) and (25).

The geometric parameter G has the particular value $G=1/4$ for circular cell material, and $G=1/2$ for lamellar cell material. It is related to the parameter $f=f_1/f_2$ appearing in Eqs. (16) and (27) by the equality (also consult the respective three-dimensional case in Ref. [33])

$$f = f_1/f_2 = \frac{4G-1}{2-4G} \quad \left(\tfrac{1}{4} \leqslant G \leqslant \tfrac{1}{2},\ 0 \leqslant f \leqslant \infty\right). \tag{32}$$

For an elliptical cell material the geometric parameter G has the particular value [24]

$$G = \tfrac{1}{2}[A^2 + (1-A)^2], \tag{33}$$

where A is the axial ratio of the elliptical cell with aspect ratio $\alpha = (1-A)/A$.

Refining a method developed by Bergman [39], Milton [26] was able to derive the bounds for two-component cell materials more restrictive than the bounds (29) and (30), which coincide with the ones from Eqs. (15) and (16) and Eqs. (26) and (27) in the two-component case [keeping in mind the relation (32), and supposing that $\lambda_2 \geqslant \lambda_1$]

$$M_\lambda^u \geqslant \lambda_c \geqslant M_\lambda^l, \tag{34}$$

$$\begin{aligned} M_\lambda^u &= \lambda_2 \frac{(\lambda_1+\lambda_2)(\lambda_1 + v_1\lambda_1 + v_2\lambda_2) - v_2[v_1 + (v_2-v_1)(4G-1)](\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)(\lambda_2 + v_2\lambda_1 + v_1\lambda_2) - v_2[v_1 + (v_2-v_1)(4G-1)](\lambda_1-\lambda_2)^2}, \\ M_\lambda^l &= \lambda_1 \frac{(\lambda_1+\lambda_2)(\lambda_2 + v_1\lambda_1 + v_2\lambda_2) - v_1[v_2 + (v_1-v_2)(4G-1)](\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)(\lambda_1 + v_2\lambda_1 + v_1\lambda_2) - v_1[v_2 + (v_1-v_2)(4G-1)](\lambda_1-\lambda_2)^2}. \end{aligned} \tag{35}$$

For the whole class of quasisymmetric two-component materials one has

$$\max_{1/4\leqslant G\leqslant 1/2} M_\lambda^u \geqslant \lambda_c \geqslant \min_{1/4\leqslant G\leqslant 1/2} M_\lambda^l. \tag{36}$$

The bounds (36) are tighter than the bounds (31) or (14) and (25). Numerical comparisons between the different bounds and the models will be illustrated in Sec. III. Note that the bounds (34) and (36) are restricted to two-component materials, while our bounds (13), (24), (17), and (28) are valid for general multicomponent materials.

III. GEOMETRIC MODELS

A natural question is how well the constructed bounds approximate the expected properties of particular composites and reflect the observed scatter intervals, given the uncertainty in composite microgeometry, and can the bounds be significantly improved? So we constructed some cell models,

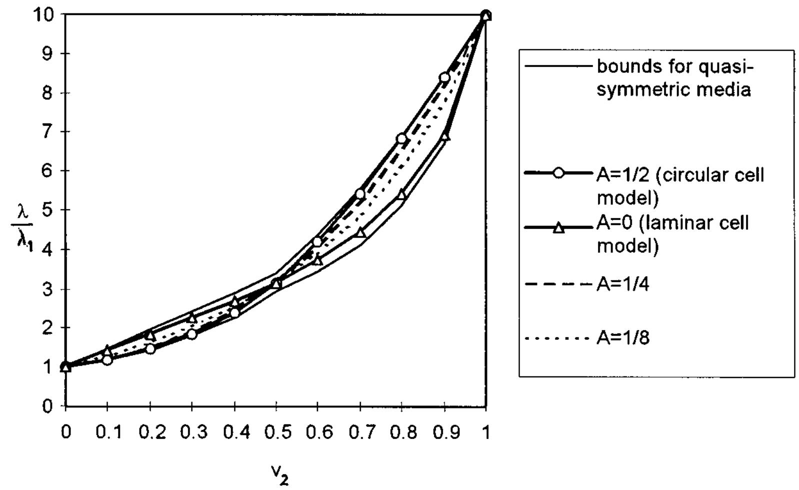

FIG. 2. Conductivity of differential elliptical cell media ($\lambda_2 = 10\lambda_1$, $v_2 = 0 \rightarrow 1$).

the effective conductivity of which could be determined exactly and compared them with the bounds.

We construct the polydispersed differential cell model following the procedure described at the end of Sec. I (for more details see Refs. [9, 37, 38]). Of particular use is the asymptotic expression for the conductivity $\lambda + d\lambda$ of a dilute suspension $v_\alpha dt$ of well-separated elliptical inclusions with the same axial ratio A from α components ($\alpha = 1, \ldots, n$) having conductivities λ_α in a matrix having conductivity λ ([38,40])

$$\lambda + d\lambda = \lambda\left[1 + \sum_{\alpha=1}^{n} v_\alpha dt \, \frac{\lambda_\alpha - \lambda}{2} \times \left(\frac{1}{\lambda_\alpha A + \lambda(1-A)} + \frac{1}{\lambda_\alpha(1-A) + \lambda A}\right)\right]. \tag{37}$$

Equation (37) leads to the differential equation of the differential scheme. In the limit with the original base matrix eliminated it yields the equation determining the effective conductivity λ_c, which exactly coincides with the equation of the effective medium self-consistent scheme

$$\sum_{\alpha=1}^{n} v_\alpha(\lambda_\alpha - \lambda_c)\left(\frac{1}{\lambda_\alpha A + \lambda_c(1-A)} + \frac{1}{\lambda_\alpha(1-A) + \lambda_c A}\right) = 0. \tag{38}$$

For circular cell media $A = 1/2$, Eq. (38) reduces to

$$\sum_{\alpha=1}^{n} v_\alpha \frac{\lambda_\alpha - \lambda_c}{\lambda_\alpha + \lambda_c} = 0. \tag{39}$$

In the case of the two-component circular cell media, Eq. (39) is resolved explicitly

$$\lambda_c = \tfrac{1}{2}\{(v_1 - v_2)(\lambda_1 - \lambda_2) + [(v_1 - v_2)^2(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2]^{1/2}\}. \tag{40}$$

Now we construct a quasisymmetric laminar cell model: form a macrocell by combining a great number of thin laminae in parallel, the conductivities of which are assigned the values λ_α randomly with the frequencies according to the volume fractions v_α of the components ($\alpha = 1, \ldots, n$). The macrocell has the conductivities $\lambda_\parallel = \Sigma_{\alpha=1}^{n} v_\alpha \lambda_\alpha$ in the direction parallel to the laminae and $\lambda_\perp = (\Sigma_{\alpha=1}^{n} v_\alpha/\lambda_\alpha)^{-1}$ in the perpendicular direction. The next step is to form a macroscopically isotropic aggregate from this base anisotropic macrocell. According to Ref. [41] the aggregate should have the effective property

$$\lambda_c = (\lambda_\parallel \lambda_\perp)^{1/2} = \left[\left(\sum_{\alpha=1}^{n} v_\alpha \lambda_\alpha\right)\left(\sum_{\alpha=1}^{n} v_\alpha/\lambda_\alpha\right)^{-1}\right]^{1/2}. \tag{41}$$

If in Eq. (38) we let $A = 0$ then we get the equation for the differential polydispersed laminar cell model. One can verify that this polydispersed laminar model possesses the same effective conductivity as that of model (41) for the two-rank laminar model constructed above.

For example, we consider quasisymmetric two-component media with $\lambda_2 = 10\lambda_1$, $v_2 = 0 \rightarrow 1$. Graphics of the effective conductivity λ_c for differential elliptical cell models calculated from Eq. (38) with $A = 1/2$ (circular cell), $A = 1/4$, $A = 1/8$, and $A = 0$ (laminar cell) are given in Fig. 2. Note that the conductivity is increased with A when $v_2 > 0.5$, while the reversed order is observed when $v_2 < 0.5$.

In Fig. 3 the bounds (14) and (25) [or equivalently Eq. (31)] and (36) for quasisymmetric composites and the conductivities of differential circular cell (40) and laminar cell (41) models are compared. The bounds (36) are more restrictive and the models lie strictly inside the bounds as expected.

In Fig. 4 the bounds for quasisymmetric materials and those for specific circular cell materials are presented together with the conductivity of the differential circular cell model, which lies strictly inside the bounds, as expected. The area surrounded by the bounds for circular cell materials covers a large part of the one defined by the bounds for more general quasisymmetric materials. The same can be said about any bounds for elliptical cell materials [keeping in mind the relations (32) and (33)], and the bounds for multicomponent materials (15), (16), (26), and (27).

It is known that in the special case $v_1 = v_2 = 1/2$ the quasisymmetric composites have the exact conductivity independent of particular cell structures [42–44]

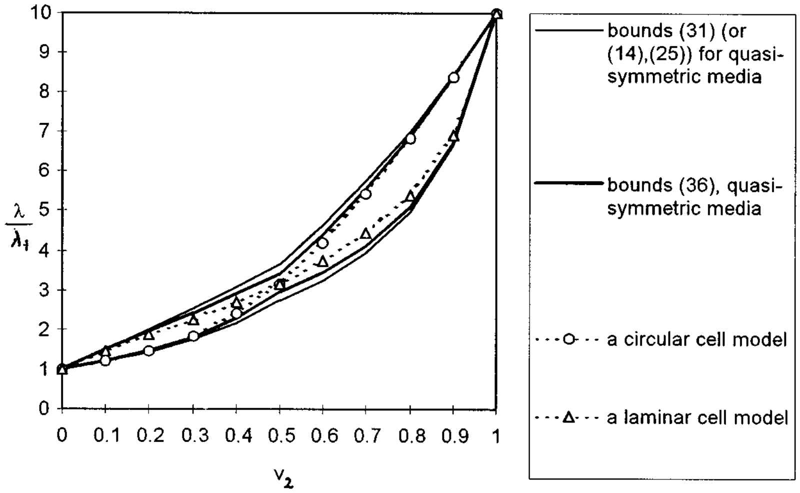

FIG. 3. Conductivity of quasisymmetric media ($\lambda_2=10\lambda_1$, $v_2=0\rightarrow 1$).

$$\lambda_c=(\lambda_1\lambda_2)^{1/2}. \tag{42}$$

So it is natural that all the cell models in Fig. 2 go through Eq. (42) at $v_1=v_2=1/2$. All the bounds, though keeping this point inside, do not converge to Eq. (42) at $v_1=v_2=1/2$. This indicates that the constructed bounds are not the best possible ones and can be improved. On the other hand, the diversity of the models in Fig. 2, outside the point $v_1=v_2=1/2$, indicates that Eq. (42) is only an exceptional case, where a property of the quasisymmetric composites can be determined exactly independent of its particular microgeometry. Generally there always exists a scatter interval for an effective property due to the uncertainty in the geometry of a quasisymmetric composite.

IV. ESTIMATES FOR THE ELASTIC MODULI

Similarly one can construct the estimates for the elastic moduli k_c, μ_c of quasisymmetric composites. The bounds for quasisymmetric two-component materials have the simple forms

$$P_K(\mu^u)\geqslant K_c\geqslant P_K(\mu^l), \tag{43}$$

$$\mu^u=\max\{v_1\mu_1+v_2\mu_2, v_1\mu_2+v_2\mu_1\},$$

$$\mu^l=\min\{(v_1/\mu_1+v_2/\mu_2)^{-1},\ (v_1/\mu_2+v_2/\mu_1)^{-1}\}, \tag{44}$$

$$P_\mu(\mu_*(K^u,\mu^u))\geqslant\mu_c\geqslant P_\mu(\mu_*(K^l,\mu^l)), \tag{45}$$

$$K^u=\max\{v_1K_1+v_2K_2, v_1K_2+v_2K_1\},$$

$$K^l=\min\{(v_1/K_1+v_2/K_2)^{-1},(v_1/K_2+v_2/K_1)^{-1}\}, \tag{46}$$

where the property functions P_K, P_μ for the general multicomponent media have forms very similar to that of P_λ in formula (6)

$$P_K(\mu_0)=\left(\sum_{\alpha=1}^{n}\frac{v_\alpha}{K_\alpha+\mu_0}\right)^{-1}-\mu_0, \tag{47}$$

$$P_\mu(\mu_*)=\left(\sum_{\alpha=1}^{n}\frac{v_\alpha}{\mu_\alpha+\mu_*}\right)^{-1}-\mu_*,$$

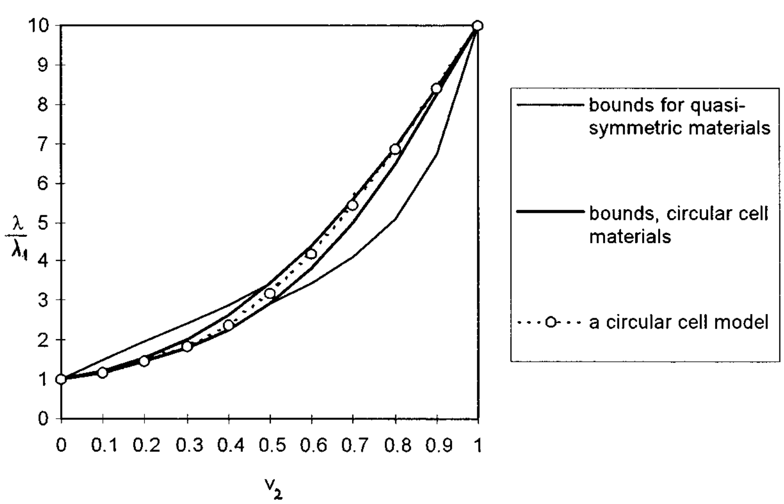

FIG. 4. Conductivity of circular cell media ($\lambda_2=10\lambda_1$, $v_2=0\rightarrow 1$).

$$\mu_*(K_0,\mu_0)=\frac{K_0\mu_0}{K_0+2\mu_0}. \tag{48}$$

In the general multicomponent case the bounds for circular cell materials are particularly simple

$$P_K(\mu^{uc})\geqslant K_c\geqslant P_K(\mu^{lc}), \tag{49}$$

$$\mu^{uc}=\sum_{\alpha=1}^{n} v_\alpha\mu_\alpha,\quad \mu^{lc}=\left(\sum_{\alpha=1}^{n} v_\alpha/\mu_\alpha\right)^{-1}, \tag{50}$$

$$P_\mu(\mu_*(K^{uc},\mu^{uc}))\geqslant\mu_c\geqslant P_\mu(\mu_*(K^{lc},\mu^{lc})), \tag{51}$$

$$K^{uc}=\sum_{\alpha=1}^{n} v_\alpha K_\alpha,\quad K^{lc}=\left(\sum_{\alpha=1}^{n} v_\alpha/K_\alpha\right)^{-1}. \tag{52}$$

Now we construct some circular and laminar cell models, the elastic moduli of which can be determined exactly. Application of the differential scheme leads to the system of equations determining the elastic moduli K_c,μ_c for the polydispersed differential circular cell model, which coincide with the respective equations of the symmetric self-consistent approximation

$$\sum_{\alpha=1}^{n} v_\alpha\frac{(K_\alpha-K_c)(K_c+\mu_c)}{K_\alpha+\mu_c}=0$$

$$\sum_{\alpha=1}^{n} v_\alpha\frac{(\mu_\alpha-\mu_c)(\mu_c+\mu_*)}{\mu_\alpha+\mu_*}=0,\quad \mu_*=\frac{K_c\mu_c}{K_c+2\mu_c}. \tag{53}$$

Next we construct some laminar cell models. We are especially interested in the extremal configurations—those that have maximal and minimal elastic moduli. Form a macrocell by combining a great number of thin laminae bonded together in parallel, the elastic moduli of which are assigned the values K_α,μ_α randomly with the frequencies according to the volume fractions v_α of the components $(\alpha=1,\ldots,n)$. The elastic coefficients c_{ijkl} of this anisotropic macrocell can be calculated following Ref. [45]

$$\begin{aligned}
c_{1111}&=\left(\sum_{\alpha=1}^{n}\frac{v_\alpha}{K_\alpha+\mu_\alpha}\right)^{-1},\\
c_{1122}&=\left(\sum_{\alpha=1}^{n}\frac{v_\alpha(K_\alpha-\mu_\alpha)}{K_\alpha+\mu_\alpha}\right)c_{1111},\\
c_{2222}&=\sum_{\alpha=1}^{n}\frac{v_\alpha[4K_\alpha\mu_\alpha+(K_\alpha-\mu_\alpha)c_{1122}]}{K_\alpha+\mu_\alpha},\\
c_{1212}&=\left(\sum_{\alpha=1}^{n}\frac{v_\alpha}{\mu_\alpha}\right)^{-1}.
\end{aligned} \tag{54}$$

Now with this base macrocell, following Refs. [27, 32], one can construct isotropic aggregates having maximal and minimal elastic moduli. In particular, the extremal laminar cell configurations constructed should have, respectively, the effective elastic moduli

$$K_c=\tfrac{1}{2}[(c_{1111}c_{2222})^{1/2}+c_{1122}]; \tag{55}$$

$$K_c=\frac{c_{1111}c_{2222}-c_{1122}^2}{c_{1111}+c_{2222}-2c_{1122}}; \tag{56}$$

$$\mu_c=\frac{c_{1111}c_{2222}-c_{1122}^2}{2c_{1122}-2c_{2222}+2\{c_{2222}[c_{1111}+c_{2222}-2c_{1122}+(c_{1111}c_{2222}-c_{1122}^2)c_{1212}^{-1}]\}^{1/2}}; \tag{57}$$

$$\mu_c=\frac{c_{1111}c_{2222}-c_{1122}^2}{2c_{1122}-2c_{1111}+2\{c_{1111}[c_{1111}+c_{2222}-2c_{1122}+(c_{1111}c_{2222}-c_{1122}^2)c_{1212}^{-1}]\}^{1/2}}. \tag{58}$$

For illustration we consider quasisymmetric two-component media with $K_1=10K_2$, $\mu_1=5K_2$, $\mu_2=0.4K_2$, and $v_1=0\to1$. Comparisons between the bounds and the models for the elastic two-dimensional bulk and shear moduli are given in Figs. 5 and 6.

Numerical and analytical comparisons reveal that the lower bound on the bulk modulus (43), (44) and the modulus of the laminar cell model (54), (56) coincide over half the ranges of parameters, in particular, when

$$v_1/\mu_2+v_2/\mu_1\geqslant v_1/\mu_1+v_2/\mu_2, \tag{59}$$

hence the lower bound on the bulk modulus of quasisymmetric two-component materials is optimal over those ranges of parameters.

Numerical comparisons show that the lower bound on μ_c (45) and (46) is nearly reached by the lower bound envelope of models (53) and (54), (57) [or (58)]. Generally the areas extended by the circular and laminar cell models, any point inside which should be attained by a specific composite, cover most of the areas bounded by the estimates for the elastic bulk and shear moduli of quasisymmetric composites (43)–(46). Hence, the constructed bounds might be near to the best possible ones determined by the uncertainty in geometry of quasisymmetric composites.

We can see that the bounds for the subclass of circular cell materials cover large parts of the areas defined by the bounds for more general quasisymmetric composites. This indicates that the very simple estimates for circular cell ma-

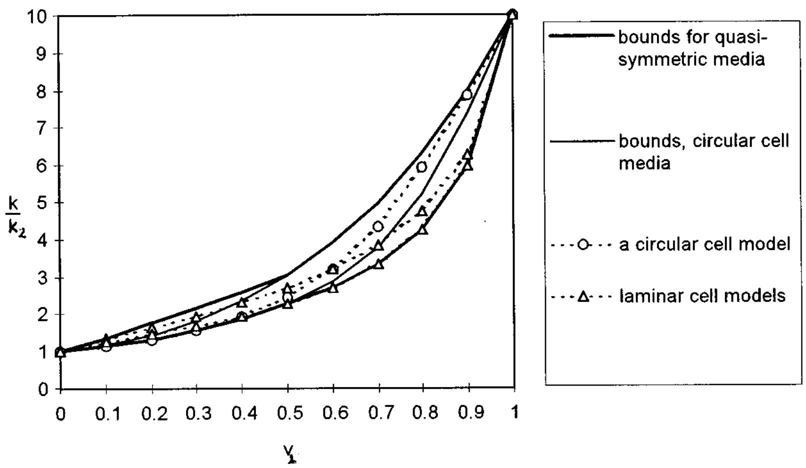

FIG. 5. Bulk modulus of quasisymmetric media $(K_1=10K_2,\ \ \mu_1=5K_2,\ \ \mu_2=0.4K_2,\ \ v_1=0\rightarrow1)$.

terials (47)–(52) can be recommended for evaluation of the properties of quasisymmetric media generally. The estimates should be best for practical equiaxial particulate composites, which are better described by the circular cell model. The estimates (47)–(52) are explicit and are much simpler than the implicit solution of the equations of the symmetric self-consistent scheme (53).

V. CONCLUSION

Most practical heterogeneous media have complicated microgeometries, hence various effective medium approximation schemes have been constructed to evaluate the effective properties of the media. Perhaps a most well-known scheme is the self-consistent one, which has been shown to be realizable by the definite though idealistic polydispersed differential model. The scheme is expected to approximate the behavior of quasisymmetric composites. However, since the microgeometry of most practical quasisymmetric composites is of random nature, there always exist certain scatter intervals in the observed effective properties of the media. Hence, the more complete approach to the problem is to predict the possible maximal and minimal values of the effective properties of the media due to the uncertainty in their microgeometry.

Upper and lower bounds on the overall elastic and conductivity properties of quasisymmetric media expressed in the properties and volume fractions of their components have been given and shown to be nearly reached by envelopes of the properties of various exact cell models, hence the constructed bounds are reasonable. The lower bound on the bulk modulus of quasisymmetric two-component composites (43), (44) is exactly attained by the laminar cell model (54), (56) over half the ranges of parameters (59).

The estimates for general quasisymmetric media do not differ much (especially around the point of equal volume fractions of phases) from those for the subclass of circular cell composites. Hence, the very simple estimates for the conductivity (17), (28) and elastic moduli (47)–(52) of multicomponent circular cell composites can be recommended for evaluation of the properties of the more general quasisymmetric composites—especially of the practical particulate ones. They are not only very much simpler than the implicit solutions of Eqs. (39) and (53) of the effective medium self-consistent approximation, but also yield certain information about the degree of scatter in the observed effective properties associated with the uncertainty in microgeometry of the composites. Observations made here also apply to three-dimensional media. More specifically, the estimates of the types (17), (28), and (47)–(52) are also valid for multicomponent spherical cell materials, which represent

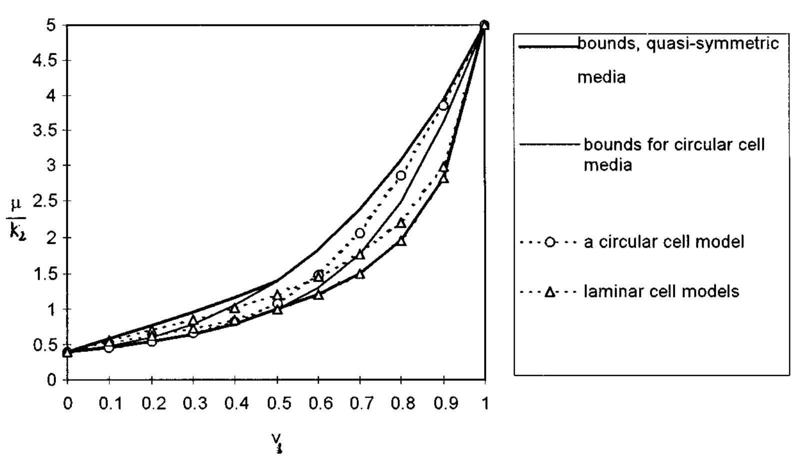

FIG. 6. Shear modulus of quasisymmetric media $(K_1=10K_2,\ \ \mu_1=5K_2,\ \ \mu_2=0.4K_2,\ \ v_1=0\rightarrow1)$.

three-dimensional particulate composites, and the only change is in the particular forms of the P functions (consult Refs. [31, 33]). The respective estimates for the quasisymmetric three-dimensional two-component materials are optimal over half the ranges of parameters [under restrictions similar to that of Eq. (59)] in three cases: the upper and lower bounds on the bulk modulus, and the upper bound on the conductivity.

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